# An Introduction to Ramsey Theory
## Honors Thesis

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**Abstract**

Ramsey theory dictates that large structures will contain ordered substructures. In this paper we present the basic Ramsey theorems and explore some of their consequences. We also discuss some known results on Ramsey numbers.

## 1 Introduction

Ramsey theory is that branch of mathematics that investigates the order inherent in any large mathematical structure. Its theorems assert that sufficiently large mathematical structures must necessarily contain substructures possessing a higher degree of order than the original structure. The essence of this powerful collection of theorems was perhaps best stated by the American mathematician T. S. Motzkin: "Complete disorder is impossible."

Ramsey theory has its origins in a 1930 paper in mathematical logic by the British mathematician Frank P. Ramsey. Frank Ramsey was an excellent mathematics student; he completed his education at Cambridge with high honors. His passions were philosophy, mathematical logic and economics. In addition to the theorem which bears his name he authored several papers in mathematical economics which, almost seventy years later, continue to be standard references in the field. Sadly, the world was deprived of his genius prematurely: Frank Ramsey died in 1930 of complications from abdominal surgery, a mere 26 years of age.

Ramsey's theorem is essentially a generalization of Dirichlet's Pigeonhole Principle to partitions of infinite sets:

**Theorem 1.1 (Pigeonhole Principle).** *Suppose we have $n$ boxes and $kn+1$ objects. Then regardless of how we place objects in boxes, some box contains at least $k+1$ objects.*

"Ramsey-type problems", i.e., problems having the flavor of Ramsey's theorem, are often simple to state and easy to understand but can be quite difficult to solve. In this paper we endeavor to present some basic results in Ramsey theory at a level of sophistication comfortable for the undergraduate. We also provide several general references to assist the more ambitious reader in further investigation of the subject.

We begin our discussion with an example.

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## 1.1 Examples of Ramsey-type problems

**Example 1.1 (Cocktail party dynamics).** It is common parlance to refer to two persons who know each other as "mutual acquaintances". Let us extend this notion to a arbitrary group of persons in the obvious fashion: we will say that the members of a group are *mutually acquainted* if every member knows every other member. On the other hand, if every member is unacquainted with every other member we will say that the members of the group are *mutually unacquainted.* To illustrate, three close friends would be mutually acquainted, while three complete strangers would be mutually unacquainted.

Suppose we wish to plan a cocktail party. A recent study by an expert on social psychology suggests that if a cocktail party is to succeed it must satisfy the following condition:

**Condition 1 (Successful Party Condition).** There are a sufficient number of persons in attendance so that at least one of the following statements is true:

1. There is a group of three persons who are mutually acquainted.

2. There is a group of three persons who are mutually unacquainted.

Assuming that we do not know who knows whom among the invitees, what is the minimum number of persons we need to invite to ensure the success of our party?

Obviously we need at least three persons, but exactly three persons will not always work: if A and B know each other but neither knows C, we fail to satisfy both criteria. Four persons is also not sufficient; in fact, neither is five. We need only show that it fails for five, since if it works for four it works for five (ignore the extra person).

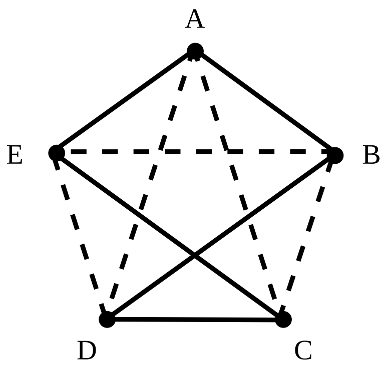


Figure 1

Consider a five-person party with the relations depicted in Figure 1: each vertex of Figure 1 corresponds to one of the attendees. A solid line corresponds to two persons who are mutually acquainted, and a dashed line corresponds to two persons who are mutually unacquainted. Clearly then, this party satisfies the Successful Party Condition if and only if the figure contains a solid triangle (corresponding to a group of three mutually acquainted persons) or a dashed triangle (corresponding to a group of three mutually unacquainted persons). Hence the Successful Party Condition fails for this arrangement.

We claim that a group of six persons will suffice. Choose one person, say A, and consider her relations to the others. There are five other persons and two classes (knowing and not knowing); by the Pigeonhole Principle, therefore, A either knows three of the others or does not know three of the others.

Suppose first that she knows three persons, say C, D and E. If some pair of them (say C and D) know each other, then A, C and D from a trio of mutually acquainted persons, and we are done. Otherwise C, D, and E form a trio of persons who are mutually unacquainted.

On the other hand, if A is mutually unacquainted with three of the others, the above argument (with "know" replaced by "does not know", etc.) shows that the Successful Party Condition is satisfied.

Thus a party with six persons will be successful, but a party with five or fewer persons will be a disaster. It should be clear, however, that we can invite *more* than six persons and still have a successful party. (Any six-person subset of them will satisfy the Successful Party Condition.)

We shall need the following definitions for the next example.

**Definition 1.1.** A **graph** $G(V, E)$ is a finite nonempty set $V$ together with a collection $E$ of two-element subsets of $V$. The elements of $V$ are called the vertices of the graph, and the elements of $E$ are called the edges of the graph.

**Definition 1.2.** Given a graph $G(V, E)$. Two vertices $v_1, v_2$ of $V$ are said to be **adjacent** if $\{v_1, v_2\} \in E$, i.e., there is an edge of $G$ connecting $v_1$ and $v_2$.

**Definition 1.3.** A graph is said to be **complete** if every two of its vertices is adjacent. More specifically, the **complete graph on** $n$ **points**, denoted $K_n$, is the graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E$ equal to the collection of all two-element subsets of $V$.

The complete graph $K_n$ is simply the graph resulting from making every possible connection between any two of the $n$ vertices.

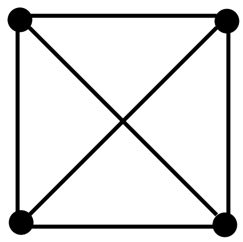

Figure 2a: $K_4$

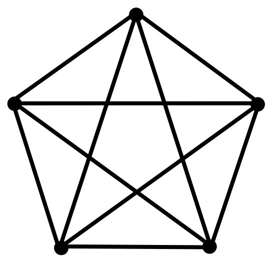

Figure 2b: $K_5$

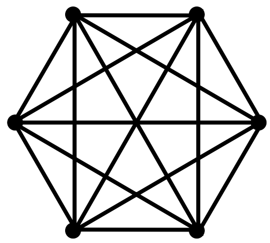

Figure 2c: $K_6$

**Definition 1.4.** Given a graph $G(V, E)$. An $r$**-coloring** of the edges of $G$ is a function $\chi : E \to \{1, 2, \ldots, r\}$.

A coloring assigns an integer (which we often think of as a "color") to each edge of $G$. Moreover, a coloring is a (disjoint) partition of $E$, the edge set. It is easy to see that a coloring defines an equivalence relation (namely, $(x_1, y_1) \sim (x_2, y_2) \iff \chi((x_1, y_1)) = \chi((x_2, y_2))$) and vice versa.

We will often use the term coloring without explicitly mentioning the associated function.

**Definition 1.5.** A subset $A \subseteq E$ is said to be **monochromatic** under the $r$-coloring $\chi$ if $\chi \restriction A$, the restriction of $\chi$ to $A$, is constant, i.e., all edges in $A$ are colored the same.

**Example 1.2 (Graph colorings).** Suppose we color every edge of $K_n$ either red or blue. What is the least number of vertices we need to ensure that no matter how we color the edges there will be either a blue triangle or a red triangle? Stated differently, what is the smallest $n$ such that any 2-coloring of $K_n$ contains a monochromatic triangle?

This problem is merely a disguised version of Example 1.1. Let the vertices of $K_n$ correspond to the invitees; then the edges of the graph correspond to the relations between them. Color an edge red if the two persons represented by its endpoints are mutually acquainted, and blue otherwise. Under this correspondence, we see that the problem of finding the smallest $n$ such that $K_n$ contains a monochromatic triangle under any 2-coloring is equivalent to our cocktail party problem.

## 1.2 The Finite Ramsey Theorem

We now begin our discussion of the Finite Ramsey Theorem. In this and all future discussions we will denote the set $\{1, 2, \ldots, n\}$ by $[n]$.

**Notation 1.** Given a set $X$, we will denote the collection of all $k$-element subsets of $X$ by $X^k$.

It is sometimes easier to view $X^k$ as the set of all $k$-tuples of elements taken from $X$, i.e., the set $\underbrace{X \times \cdots \times X}_{k \text{ times}}$. When making this identification, however, it is important to realize that we are not concerned with the order of the elements in the $k$-tuple. Hence in our discussions, for example, $(1, 0)$ and $(0, 1)$ are considered the same.

**Definition 1.6.** An **$r$-coloring** of $[n]^k$ is a function $\chi : [n]^k \to [r]$.

**Definition 1.7.** A set $A \subseteq [n]$ is said to be **monochromatic** under a coloring $\chi$ of $[n]^k$ if the function $\chi \upharpoonright A^k$ is constant, i.e, all $k$-tuples of $A^k$ are colored the same.

For $k = 2$ this is agrees with our earlier definition of monochromatic (Definition 1.5).

**Theorem 1.2 (Finite Ramsey Theorem, First Formulation).** *For all integers $k$,$r$, $l_1, \ldots, l_r$ there exists an integer $n_0$ such that if $n \geq n_0$ and $\chi$ is an $r$-coloring of $[n]^k$ then there exists an integer $1 \leq i \leq r$ and a subset $A$ of $[n]$ such that $|A| = l_i$ and the function $\chi \upharpoonright A^k \equiv i$, i.e., every $k$-tuple of $A^k$ is colored $i$.*

*Proof.* The Finite Ramsey Theorem can be proven directly using induction. However, it is much easier to deduce it from the Infinite Ramsey Theorem, to be discussed below. $\square$

We now introduce the arrow notation of Erdös and Rado as a more compact means of stating Ramsey-type theorems.

**Notation 2.** Given integers $n$, $r$, $k$, $l_1, \ldots, l_r$ we write

$$n \mapsto (l_1, \ldots, l_r)^k$$

if, for every $r$-coloring of $[n]^k$, there exists an integer $i$, $1 \leq i \leq r$, and a set $A \subset [n]$ such that $|A| = l_i$ and every $k$-tuple of $[A]^k$ is colored $i$. In the case that $l_1 = \cdots = l_r = l$ we will use the simpler notation $n \mapsto (l)^k_r$. Furthermore, if either the dimension $k$ or the number of colors $r$ is not stated it is assumed to be 2. Thus $6 \mapsto (3)$ and $6 \mapsto (3)^2_2$ are equivalent to $6 \mapsto (3, 3)^2$.

The first formulation of the Finite Ramsey Theorem can thus be restated more succinctly as follows.

**Theorem 1.3 (Finite Ramsey's Theorem, Second Formulation).** *For all integers $k$, $l_1, \ldots, l_r$ there exists an integer $n_0$ so that whenever $n \geq n_0$ we have $n \mapsto (l_1, \ldots, l_r)^k$*

The **Ramsey function** $R_k(l_1, \ldots, l_r)$ is defined to be the smallest integer $n$ such that $n \mapsto (l_1, \ldots, l_r)^k$. The Finite Ramsey Theorem essentially guarantees that the Ramsey function is well-defined for all integers $k$, $l_1, \ldots, l_r$.

The values of the Ramsey function are called **Ramsey numbers**. There are some common abbreviations for Ramsey numbers when symmetry is involved. If $l_1 = \cdots = l_r = l$ we write

$$R_k(l_1, \ldots, l_r) = R_k(l; r)$$

or the more compact

$$R_k(l_1, \ldots, l_r) = R_k(l)$$

if $r = 2$. When $k = 2$ we usually omit the "$k$" from the notation. Hence, for instance, $R(5)$ is shorthand for $R_2(5, 5)$.

## 1.3 Examples of Ramsey-Type Problems

It should be clear that examples 1.1 and 1.2 both follow from the fact that $6 \mapsto (3)$.

**Example 1.3.** We make the following observations ([GRS]):

1. If $n \mapsto (l_1, \ldots, l_r)^k$ and $m \geq n$, then $m \mapsto (l_1, \ldots, l_r)^k$, for we may ignore the extra $m - n$ elements;
2. If $n \mapsto (l_1, \ldots, l_r)^k$ and $l'_i \leq l_i$, $1 \leq i \leq r$, then $n \mapsto (l'_1, \ldots, l'_r)^k$, for we may delete the extra elements;
3. If $n \mapsto (l_1, \ldots, l_r)^k$ and $\sigma$ is a permutation of $[n]$, then $n \mapsto (l_{\sigma(1)}, \ldots, l_{\sigma(r)})$, for $\sigma \cdot \chi$ is also an $r$-coloring of $[n]$.

**Example 1.4.** For $r = 1$ the Finite Ramsey Theorem is trivial. For $k = 1$ and $l_1 = \cdots = l_r = l$ it states that for all $n \geq R(l; r)$ and any $r$-coloring of $[n]$ there exists a monochromatic $l$-element subset of $[n]$. The reader will recognize this as the usual Pigeonhole Principle.

**Example 1.5.** Any sequence of distinct reals of length at least $R(n+1)$ contains a monotone subsequence of length $n+1$ ([GRS]).

*Proof.* Given a sequence $\{b_i\}$, $1 \leq i \leq R(n+1)$, we consider the following 2-coloring. For any two indices $i$ and $j$ with $i < j$, color $(i, j)$ red if $b_i < b_j$, and blue otherwise. By Ramsey's Theorem, we know that there is either a set of $n+1$ indices whose pairs are colored red or a set whose pairs are colored blue. In either case, the set is well-ordered, so we can use the least index to build the desired sequence. $\square$

In fact, our bound is quite weak: Paul Erdös and George Szekeres showed that any sequence of length $n^2+1$ contains a monotone subsequence of length $n+1$ ([ES]).

**Proposition. 1 (Erdös-Szekeres).** *Any sequence of distinct reals of length at least $n^2+1$ contains a monotone subsequence of length $n+1$.*

*Proof.* Consider the sequence $\{x_k\} = x_1, \ldots, x_{n^2+1}$. For each element $x_j$ of the sequence define $a_j$ to be the length of the longest increasing subsequence of $\{x_k\}$ ending at $x_j$ and $b_j$ to be the length of the longest decreasing subsequence ending at $x_j$. If $\{x_k\}$ does not have a monotone subsequence of length $n+1$, then $a_j \leq n$ and $b_j \leq n$ for all $j$. Hence there are $n$ possibilities for $a_j$ and $n$ possibilities for $b_j$, so there are $n^2$ possible pairs $(a_j, b_j)$ for each $x_j$. By the Pigeonhole Principle, two elements of $\{x_k\}$ are associated with the same pair $(a^*_j, b^*_j)$. This is impossible, however, since we assumed that the elements of $\{x_k\}$ were distinct. Hence $\{x_k\}$ contains a monotone sequence of length $n+1$. $\square$

As we will see, the bounds furnished by Ramsey-like statements are often unnecessarily large.

**Example 1.6 (due to Esther Klein).** For any $n$ there exists an $N$ such that for any $N$ points in the plane we can find $n$ that form a convex $n$-gon ([WES]).

We need the following lemmas.

**Lemma 1.1.** *Given any five points in general position in the plane (that is, no three of them are collinear), some four of them determine a convex quadrilateral.*

*Proof.* Recall that the **convex hull** of a set of points is the smallest convex set containing all the points.

Construct the convex hull of the five points. There are three possibilites: a pentagon, a quadrilateral and a triangle. Clearly in the first two cases the claim holds, so without loss of generality we can assume the convex hull is a triangle.

The two points in the interior of the triangle determine a line $L$. By the Pigeonhole Principle two of the three vertices are on the same side of $L$. Then these two points, together with the two points on $L$, form a convex quadrilateral. $\square$

**Lemma 1.2.** *Given $n$ points in general position in the plane, if every four-element subset of the $n$ points forms a convex quadrilateral, then the $n$ points form a convex $n$-gon.*

*Proof.* It is easier to prove the contrapositive of the claim. Suppose that we are given $n$ points that do not determine a convex $n$-gon. We must find four points that do not determine a convex quadrilateral.

Since the $n$ points do not determine a convex $n$-gon, their convex hull is polygon with fewer sides, say $p$. The other $n-p$ points therefore lie in the interior of the $p$-gon. Fix a vertex $V$ of the $p$-gon and draw all the diagonals from $V$. This construction gives us $p-2$ triangles. Since the $n$ points are assumed to be in general position, no three of them are collinear. Therefore, none of the interior points is contained in a diagonal emanating from $P$. Hence some triangle contains a point in its interior, and this point and the vertices of the triangle determine a quadrilateral that is not convex. $\square$

With these tools in hand the proof of our original claim becomes quite easy.

**Proposition. 2 (Klein).** *For any $n$ there exists an $N$ such that for any $N$ points in the plane we can find $n$ of them that are vertices of a convex $n$-gon.*

*Proof.* Take $N = R_4(n, 5)$. Color $[N]^4$ (i.e. four-element subsets of $[N]$) by convexity: if the four points form a convex quadrilateral, we color the 4-tuple red; otherwise, we color it blue. Then by the Finite Ramsey Theory, either there exists an $n$-element subset of the $N$ points whose 4-tuples are all colored red, or there exists a 5-element subset whose 4-tuples are all blue. But Lemma 1.1 tells us that latter subset cannot exist. Therefore, we know that there is a set of $n$ points satisfying the hypothesis of Lemma 1.2, so we conclude that these $n$ points form a convex $n$-gon. $\square$

Esther Klein conjectured this result after discovering the $n = 4$, $N = 5$ case. She worked with Erdös and Szekeres[1] on its solution, which was published in their monumental 1935 paper [ES]. In the proof of the theorem, Szekeres actually proved a version of the Finite Ramsey Theorem, which, unknown to him at the time, had been published five years earlier. It was their paper, however, that led to the popularization of Ramsey theory in combinatorics.

It is still an open problem to determine an exact formula for $N(n)$, the minimum number of points needed to guarantee the existence of a convex $n$-gon, as a function of $n$. Erdös, Klein and Szekeres conjectured that $N(n) = 2^{n-2} - 1$. They managed to show that

$$2^{n-2} - 1 \le N(n) \le \binom{2n-4}{n-2} + 1.$$

Our bound, $R_4(n, 5)$, is far larger than necessary. For instance, it can be demonstrated that $N(5) = 9$, while the determination of the exact value of $R_4(5, 5)$ is a difficult and currently unsolved problem. (As of this writing the best known lower bound is $R_4(5, 5) \ge 34$, due to G. Exoo. ([RAD]))

**Example 1.7.** We recall the definition of a semigroup.

**Definition 1.8.** A **semigroup** $(S, \diamond)$ is a set $S$ with a binary operation $\diamond$ satisfying the following axioms.

| | | |
|---|---|---|
| 1. | $a \diamond b \in S$; | (closure) |
| 2. | $a \diamond (b \diamond c) = (a \diamond b) \diamond c$ | (associativity) |

for all $a, b, c \in S$.

Clearly, every group is a semigroup. The sets $\mathbb{N}, \mathbb{Q}$, and $\mathbb{R}$ are all semigroups under (ordinary) multiplication; they are examples of semigroups that are not groups (not every element has a multiplicative inverse). When there is no ambiguity we will write $ab$ to mean $a \diamond b$.

Using the Finite Ramsey's Theorem, we can prove the following result.

[1] Erdös affectionately referred to this theorem as the "Happy End Theorem", because Klein and Szekeres were wed the following year ([GC]).

**Proposition. 3.** *Every finite semigroup $S$ contains an element $x$ such that $x^2 = x$ ([GRA]). (Such an element is called an* ***idempotent****.)*

Every group contains exactly one idempotent, namely, its unique identity element $e$. (The converse, however, is not true: The semigroup $\mathbb{N}$ contains exactly one idempotent but is not a group.)

*Proof.* Let $s = |S|$. The case $s = 1$ is trivial. We first consider the case $s = 2$. In this case, it is easy to show that $S$ is actually a group, so by the previous remark, the claim holds. So we now assume $s \geq 3$.

Let $t \geq R(s; s)$. Let $(x_0, x_1, \ldots, x_t)$ be a sequence of elements of $S$, where repetition is allowed. Define an $s$-coloring $\chi$ on $K_t$ (the complete graph on t points) by

$$\chi(\{i,j\}) = \prod_{i<\alpha\leq j} x_\alpha \tag{1}$$

By the Finite Ramsey's Theorem, $K_t$ contains a monochromatic $K_s$, so clearly it contains a monochromatic triangle. Let $i, j$, and $k$ be the vertices of this triangle; then we have $\chi(\{i,j\}) = \chi(\{i,k\}) = \chi(\{j,k\})$, that is,

$$\prod_{i<\alpha\leq j} x_\alpha = \prod_{i<\alpha\leq k} x_\alpha = \prod_{j<\alpha\leq k} x_\alpha \tag{2}$$

Take $x = \prod_{i<\alpha\leq k} x_\alpha$. Then

$$x^2 = \left(\prod_{i<\alpha\leq k} x_\alpha\right)\left(\prod_{i<\alpha\leq k} x_\alpha\right) = \left(\prod_{i<\alpha\leq j} x_\alpha\right)\left(\prod_{j<\alpha\leq k} x_\alpha\right) = \left(\prod_{i<\alpha\leq j} x_\alpha\right) = x.$$

□

## 1.4 The Infinite Ramsey Theorem

We now turn to the Infinite Ramsey Theorem, the version originally formulated by Ramsey in his 1930 paper [RAM].

**Theorem 1.4 (Infinite Ramsey's Theorem).** *Let $A$ be an infinite set, $k$ a natural number, and $\chi : A^k \to [r]$ be a $r$-coloring of the $k$-tuples of $A$ (or equivalently, the elements of $A^k$). Then $A$ contains an infinite set $B$ such that $\chi \restriction B^k$ is constant, that is, every $k$-tuple of elements of $B$ is colored the same color.*

The Infinite Ramsey Theorem can be stated concisely in arrow notation: $\aleph_0 \mapsto (\aleph_0)^k_r$, where $\aleph_0$ is the first infinite cardinal (and the cardinality of $\mathbb{N}$).

We will follow the proof of ([BOL]).

*Proof.* We apply strong induction on $k$. For $k = 1$, the claim is clearly true: we are partitioning an infinite set into finitely many classes, so some class must contain infinitely many elements. Assume then that $k > 1$ and the claim holds for all smaller values of $k$.

Our strategy is to use our inductive hypothesis to construct a decreasing sequence of monochromatic sets (not necessarily all the same color) and use them to find an infinite sequence of points that is monochromatic under $\chi$.

Let $A_0 = A$ and choose an element $x_0 \in A_0$. Define $B_1 = A_0 - \{x_0\}$. We define a $r$-coloring $\chi_1$ on $(k-1)$-tuples of $B_1$ by

$$\begin{aligned} \chi_1 : B_1^{(k-1)} &\to [r] \\ (b_{1,1}, \ldots, b_{1,k-1}) &\mapsto \chi(b_{1,1}, \ldots, b_{1,k-1}, x_0) \end{aligned}$$

By our induction hypothesis, there exists an infinite set $A_1 \subset B_1 \subset A_0$ such that $\chi_1$ is constant on $A_1^{(k-1)}$, i.e., the $(k-1)$-tuples of $A_1$ are all the same color, say $d_1 \in [r]$.

Choose an arbitrary element $x_1 \in A_1$, and define $B_2 = A_1 - \{x_1\}$. Define a $r$-coloring $\chi_2$ of $(k-1)$-tuples of $B_2$ by $\chi_2((b_{2,1}, \ldots, b_{2,k-1})) \mapsto \chi(b_{2,1}, \ldots, b_{2,k-1}, x_1)$.

By our induction hypothesis, there exists an infinite set $A_2 \subset B_2 \subset A_1$ such that the $(k-1)$-tuples of $A_2$ are all the same color $d_2$.

Continue inductively in this fashion, defining

$$\begin{aligned} x_n &\in A_n \\ B_{n+1} &= A_n - \{x_n\} \\ \chi_{n+1}((b_{n+1,1}, \ldots, b_{n+1,k-1})) &= \chi(b_{n+1,1}, \ldots, b_{n+1,k-1}, x_n) \end{aligned}$$

Using the induction hypothesis, there exists an infinite set $A_n \subset B_n$ such that its $(k-1)$-tuples are all the same color $d_n$. We obtain a decreasing sequence of sets

$$A_0 \supset B_1 \supset A_1 \supset \cdots \supset B_n \supset A_n \ldots$$

and corresponding sequence of elements $x_i \in A_i$ and colors $d_i$. Furthermore, for any positive integer $i$, let $\bar{x}$ be an $k$-tuple of $A^k$ whose only element outside $A_i$ is $x_{i-1}$. Then the $(k-1)$-tuple formed by omitting $x_{i-1}$ is an element of $A_i^{(k-1)}$, so it has color $d_i$. Hence by our definition of $\chi_i$ (the coloring acting on $A_i$) we have $\chi(\bar{x}) = d_i$, i.e. $\bar{x}$ is colored $d_i$.

Since the elements of the infinite sequence $\{d_n\}_1^\infty$ come from the finite set $[r]$, some element $d$ appears infinitely often, so we can choose a constant subsequence $\{d_{n_j}\}_1^\infty$ (where every element is equal to $d$). Define $\Omega$ to be $\{x_n : n = n_j \text{ for some } j \in \mathbb{N}\}$, i.e., the subset of the $x_n$'s whose indices correspond to the $d_{n_j}$'s. If $\bar{x} = \left(x_{n_{j_1}}, x_{n_{j_2}} \ldots, x_{n_{j_k}}\right)$ is an $k$-tuple of $\Omega^k$, its only element outside $A_{n_{j_2}}$ is $x_{n_{j_1}}$, so $\chi(\bar{x}) = d_{n_{j_2}} = d$. Hence $\Omega$ is the desired monochromatic infinite set, and we are done. □

## 1.5 More Examples of Ramsey-Type Problems

**Example 1.8.** A **partial order** on a set X is a relation $\leq$ satisfying the following three axioms.

1. (Reflexivity) $x \leq x$ for all $x \in X$;
2. (Antisymmetry) $x \leq y$ and $y \leq x \implies x = y$ for all $x, y \in X$;
3. (Transitivity) $x \leq y$ and $y \leq z \implies x \leq z$ for all $x, y$ and $z \in X$.

We write $(X, \leq)$ for the partially ordered set and frequently abbreviate "partially ordered set" to "poset".

We say that two elements $x, y$ of a poset are **comparable** if either $x \leq y$ or $y \leq x$. If neither relation holds, $x$ and $y$ are said to be **incomparable**. If every two elements in a poset are comparable, the poset is said to form a **chain**. If, on the other hand, every pair of elements is incomparable, the poset is called an **antichain**.

Using the Infinite Ramsey's Theorem we can prove the following.

**Proposition. 4.** *Let $P$ be an infinite set and let $\leq$ be a partial ordering on $P$. Then $P$ contains an infinite sequence $\langle p_n : n \in \mathbb{N}\rangle$ that is*

1. *strictly increasing: $p_1 < p_2 < \ldots$ or*
2. *strictly decreasing: $p_1 > p_2 > \ldots$ or*
3. *an antichain.*

*Proof.* We 2-color $P^2$ as follows. Color a pair $\{p,q\}$ red if $p$ and $q$ are comparable, and blue otherwise. By the Infinite Ramsey's Theorem, there exists a monochromatic infinite set $Q$. If $Q$ is blue, every pair of elements in $Q$ is incomparable, so any sequence in $Q$ forms an antichain. Otherwise, $Q$ is red, so it is an infinite chain.

Suppose $Q$ is well-ordered, i.e. every nonempty subset has a least element. Then we construct a strictly increasing sequence in the obvious fashion: $Q$ is nonempty, so it has a least element $q_1$. Then $Q - q_1$ is nonempty, so it has a least element $q_2$ which by construction satisfies $q_1 < q_2$. (If $q_1 = q_2$ then the poset $Q$ would have more than one least element, a contradiction.) Continuing this process inductively we obtain the desired strictly increasing infinite sequence.

If $Q$ is not well-ordered, some subset $R$ has no least element. Clearly $R$ is infinite, and given any $r_i \in R$ we can always find an $r_{i+1} \in R$ such that $r_i > r_{i+1}$. Hence, we can extract a strictly decreasing infinite sequence from $Q$. This completes the proof. □

**Example 1.9.** Let $G(V,E)$ be a graph with $|V| = \aleph_0$. We claim that either there exists an infinite set $A \subset V$ such that $\{v_i, v_j\} \in E$ for every distinct $v_i, v_j \in A$, or there exists an infinite set $B \subset V$ such that $\{v_i, v_j\} \notin E$ for every distinct $v_i, v_j \in B$.

*Proof.* Color a pair $(v_i, v_j)$ (where $v_i$ and $v_j$ are distinct) blue if $\{v_i, v_j\} \in E$ and red otherwise. By the Infinite Ramsey's Theory, there exists either an infinite subset $A$ such that $A^2$ is blue or an infinite subset $B$ such that $B^2$ is red, which implies our claim. □

### 1.6 The Compactness Principle

The Compactness Principle is a very powerful tool in Ramsey Theory. It is used to deduce facts about finite sets from known results about infinite sets. We will only state a version of the theorem for infinite sets of positive integers. For the statement of the more general version and its proof we refer the reader to [GRS].

**Theorem 1.5 (Compactness Principle for $\mathbb{N}$).** *Let $k$ be a fixed integer and let $F$ be a family of finite subsets of $\mathbb{N}$. Suppose that for any finite coloring $\chi$ of $[\mathbb{N}]^k$ there exists a subset $A \in F$ such that $[A]^k$ is monochromatic under $\chi$. Then for all $r \in \mathbb{N}$ there exists an integer $n_0(r)$ such that if $n \geq n_0(r)$ and $[n]^k$ is $r$-colored there exists a set $I \in F$, $I \subseteq [n]$ such that $I^k$ is monochromatic under $\chi$.*

We can now use the Compactness Principle to deduce a special case of the Finite Ramsey's Theorem from its infinite analogue.

**Corollary 1 (Ramsey's Theorem for $l_1 = \cdots = l_r = l$).** *For all integers $l, r, k$ there exists an integer $n_0(r)$ such that whenever $n \geq n_0$ we have $n \mapsto (l)^k_r$.*

*Proof.* Let $\chi$ be an $r$-coloring of $\mathbb{N}^k$. By the Infinite Ramsey's Theorem there exists an infinite subset $A$ such that $[A]^k$ is monochromatic under $\chi$. Let $F$ be the family of all $l$-element subsets of $A$. Clearly any element of $F$ is monochromatic under $\chi$, so by the Compactness Principle there exists an integer $n_0(r)$ such that if $n \geq n_0(r)$ and $[n]^k$ is $r$-colored there is an $l$-element set $I \in F$, $I \subseteq [n]$ such that $I^k$ is monochromatic under $\chi$. Equivalently, we have $n \mapsto (l)^k_r$, so the proof is complete. □

## 2 Ramsey Numbers

Relatively few Ramsey numbers are known exactly. The obvious method of calculating Ramsey numbers, i.e, testing all possible colorings on complete graphs $K_n$, requires an enormous amount of work. (To be precise, the amount of computation needed by this methods grows exponentially in $n$, the number of vertices.) Most of the current research in Ramsey theory revolves around improving known bounds for Ramsey numbers and developing better methods of calculating exact values.

The easiest Ramsey numbers to deal with are those with $k = r = 2$. The inductive proof of the Finite Ramsey's Theorem yields the recursive upper bound

$$R(n,m) \leq R(n-1,m) + R(n,m-1) \tag{3}$$

which in turn yields the weak bound

$$R(n,m) \leq \binom{n+m-2}{n-1}. \tag{4}$$

The **diagonal Ramsey numbers** $R(n) = R(n,n)$ are slightly easier to deal with due the symmetry involved. From Equation (4) we have the upper bound

$$R(n) \leq \binom{2n-2}{n-1}.$$

Erdös (1947) established the lower bound

$$\frac{1}{e\sqrt{2}} n 2^{n/2} < R(n)$$

using a clever counting argument on $K_n$. (See [GC] for the details.) However, progress towards improving these bounds has been slow, with only minor improvements in the associated constants.

For "off-diagonal" Ramsey numbers $R(m,n)$ $(m \neq n)$ the situation is not quite so neat. For large $n$, there are general asympotic upper and lower bounds for $R(m,n)$ (the lower bound is due to Spencer ([GC]), the upper bound is due to Li and Rousseau ([GC])) that hold for any fixed $m$, and slightly better bounds for specific $m$.

For **multicolor Ramsey numbers**, i.e those with $r > 2$, the inductive proof of the Finite Ramsey's Theorem yields the generalization of Equation (3), namely

$$R(l_1,\ldots,l_r) \leq 2 + \sum_{i=1}^{r} \left[R(l_1,\ldots,l_{i-1},l_i - 1,l_{i+1},\ldots,l_r) - 1\right].$$

The only nontrivial multicolor Ramsey number known exactly is $R(3,3,3) = 17$, due to Greenwood and Gleason [RAD].

Very few Ramsey numbers are known exactly for $k > 2$. (These are the so-called **hypergraph Ramsey numbers**.) The computations involved are much harder than those for the $k = 2$ case.

The interested reader may find extensive information on known values and bounds in Radziszowski's paper [RAD].

# 3 Closing Words

## 3.1 Generalizations of Ramsey's Theorem

The Infinite Ramsey's Theorem can in fact be extended to sets of arbitrary size. Erdös and Rado (1952) proved the following unexpected result.

**Theorem 3.1 (Erdös-Rado).** *For every positive integer $p$ and infinite cardinals $\alpha, \beta$ there exists a cardinal $\gamma$ such that $\gamma \mapsto (\alpha)^p_\beta$ ([NES]).*

This led to the development of the *partition calculus* of cardinal numbers. Further exploration of this subject, however, requires more machinery than we care to develop here. The monograph of Erdös, Hajnal, Máté and Rado [EHMR] is perhaps the best text on the subject; we encourage the interested reader to consult it for further details.

### 3.2 Solving Problems with Ramsey's Theorem

Effective use of Ramsey theory in proofs often hinges upon finding a "good" coloring. We are often looking for a set having a specific property and satisfying some minimum size condition. Ramsey theory, in effect, assures us that if we consider a sufficiently large set it will contain a monochromatic subset of our desired size; it makes no guarantee, however, that this set has the color (i.e., the property) we want. It is up to us to demonstrate that the sets having the undesired colors lead to logical contradictions.

This process of disproving the undesired cases usually requires additional hypotheses about the underlying sets. A "good" coloring is one that minimizes the amount of additional work we must do to rule out the unwanted potentialities. An excellent example of this technique is our proof of Esther Klein's assertion (Example 1.6), where we used our prior knowledge about convex sets to show that only the desired case could hold.

Another important general caveat is that Ramsey-type theorems often provide easy existence arguments that yield poor bounds on the data involved. We noted this in the discussions of Examples 1.5 and 1.6. This is due to the exponential growth of the associated Ramsey functions. Often these bounds can be improved significantly using clever combinatorial or probabilistic arguments, but such arguments are often far from obvious.

## 4 Further Reading

Perhaps the best general reference on the subject (and most accessible to the undergraduate) is [GRS]. The text [NES] is an excellent survey of modern results in Ramsey theory and is written at a more advanced level than [GRS]. Most books on combinatorics or graph theory devote a few pages to basic Ramsey theory. Advanced texts in set theory (such as [JEC2] and [KM]) usually discuss Ramsey theory under the heading "Infinitary Combinatorics" and explore extensions of Ramsey-type theorems to arbitrary cardinals and ordinals. For Ramsey numbers, the best source of up-to-date information is the paper of [RAD].

## References


[BOL] Béla Bollobás. *Modern Graph Theory*, Graduate Texts in Mathematics **184**, Springer-Verlag, New York-Berlin-Heidelberg, 1998.

[EHMR] Paul Erdös, A. Hajnal, A. Máté, R. Rado. *Combinatorial Set Theory: Partition Relations for Cardinals*, Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1984.

[ES] Paul Erdös and George Szekeres. *A Combinatorial Problem in Geometry*, Compositio Math. **2** (1935), 463-470.

[GOL] Robert Goldblatt. *Lectures on the Hyperreals: Introduction to Nonstandard Analysis*, Graduate Texts in Mathematics **188**, Springer-Verlag, New York-Berlin-Heidelberg, 1998.

[GC] Ronald Graham and Fan Chung. *Erdös on Graphs*, A.K. Peters, Wellesley, 1998.

[GRA] Ronald Graham. *Rudiments of Ramsey Theory*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, Number 45, American Mathematical Society, Providence, 1980.

[GRS] Ronald Graham, Bruce Rothschild and Joel Spencer. *Ramsey Theory*, Wiley Interscience Series in Discrete Mathematics, John Wiley & Sons, Inc., New York-Chichester-Brisbane-Toronto, 1980.

[GS] Ronald Graham and Joel Spencer. *Ramsey Theory*, Scientific American, July 1990, 112-117.

[JEC1] Thomas Jech and Karel Hrbacek. *Introduction to Set Theory*, Monographs and Textbooks in Pure and Applied Mathematics **85**, Marcel-Dekker, New York-Basel, 1984.

[JEC2] Thomas Jech. *Set Theory*, Pure and Applied Mathematics Series, Academic Press, New York, 1978.

[KM] K. Kuratowski and A. Mostowski. *Set Theory with an Introduction to Descriptive Set Theory*, Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1976.

[NES] J. Nesetril and V. Rödl, editors. *Mathematics of Ramsey Theory*, Algorithms and Combinatorics **5**, Springer-Verlag, New York-Berlin-Heidelberg, 1991.

[RAD] Stansilaw P. Radziszowski. *Small Ramsey Numbers*, Elec. Jour. Comb. **1** (1994), Dynamic Survey 1. URL `http://www.combinatorics.org/`.

[RAM] Frank P. Ramsey. *On a Problem of Formal Logic*, Proc. London Math. Soc. Ser. 2 **30** (1930) Part 4, 338-384.

[WES] Douglas B. West. *Introduction to Graph Theory*, Prentice-Hall, Upper Saddle River, New Jersey, 1996.